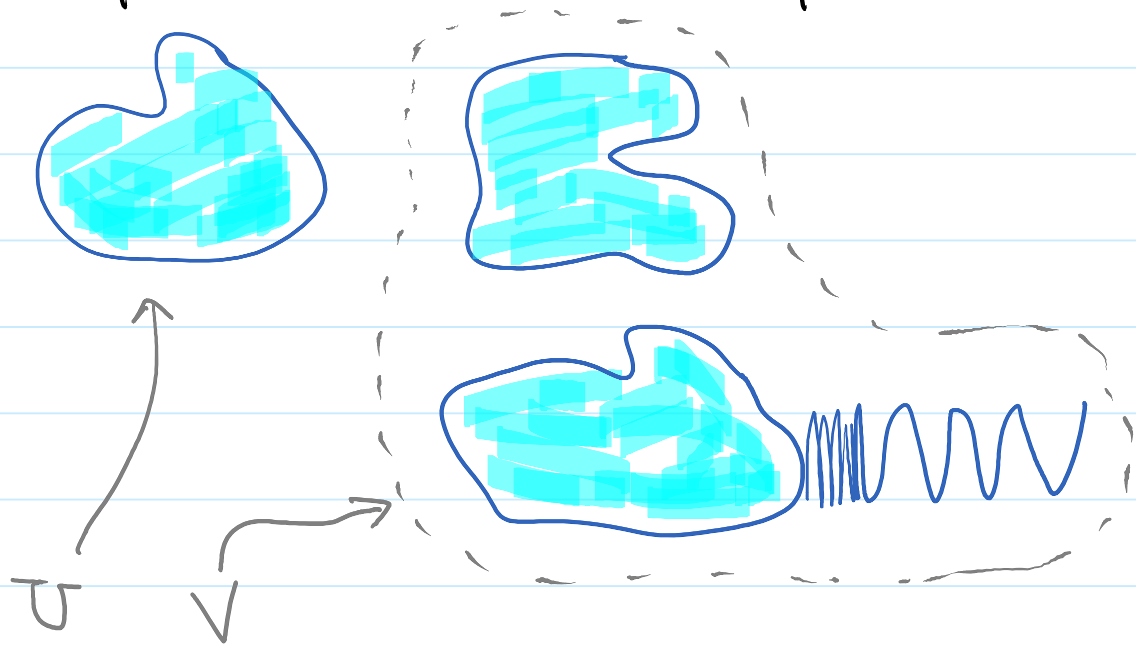
April 11, 2017

Tuesday, April 11, 2017 11:06 AM

An Intuition

Suppose X has several connected components as in the picture



So, X is disconnected and we have

$$X = U \cup V \quad \text{as above}$$

where U, V are both open & closed in X

U is already a connected component

$V = V_1 \cup V_2$, into two both open & closed connected components

Apparently, every connected component is both open and closed in X.

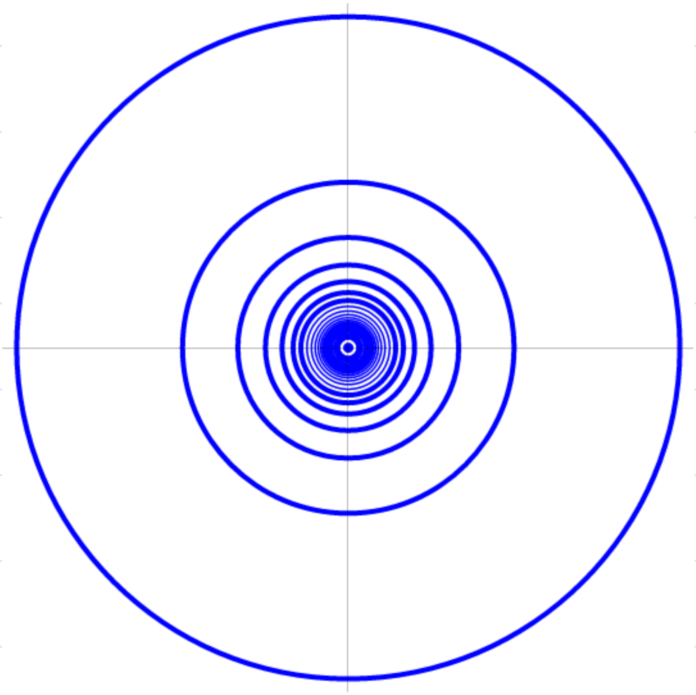
But, wrong correct

Example. $X \subset \mathbb{R}^2$ with standard topology

$X = C_0 \cup C_n$ where $C_0 = \{(0,0)\}$,

$C_n = \left\{(x,y): x^2+y^2 = \frac{1}{n^2}\right\}$ $1 \leq n \in \mathbb{N}$

Each component C_n, $n \geq 1$, is both open & closed; but C_0 is not open.



Exercise. A connected component is closed.

Theorem If X and Y are connected then so is $X \times Y$.

Inductively, a finite product of connected spaces is also connected.

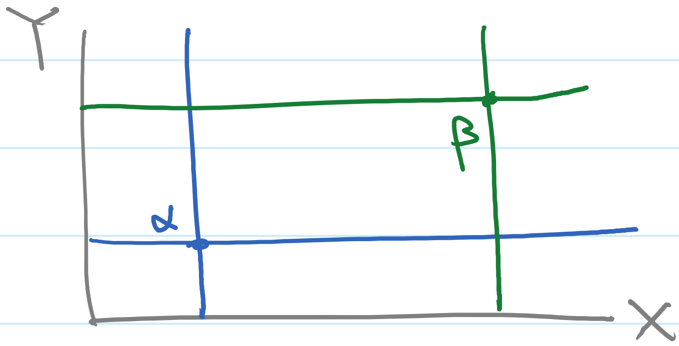
Idea of proof. Observe that for each $(a,b) \in X \times Y$,

$X \times \{b\}, \{a\} \times Y$ are connected

They intersect at $\{(a,b)\}$ and so their union is connected.

Let $A_\alpha = (X \times \{b\}) \cup (\{a\} \times Y)$, $\alpha = (a,b) \in X \times Y$

Note that $A_\alpha \cap A_\beta \neq \emptyset$ for α, β



Therefore, $X \times Y = \bigcup_\alpha A_\alpha$ is connected

Question. Will the above result or argument work for infinite product?

Let us first look at another property.

Theorem. Let $A \subset X$ be a connected set. If $A \subset B \subset \bar{A}$ then B is connected.

Proof. Let $S \subset B$ be both open and closed in B. Thus, $\exists\ G, F \subset X$ such that $G \in \mathcal{J}_X$, $X \setminus F \in \mathcal{J}_X$ and

$$S = G \cap B = F \cap B$$

$$\therefore S \cap A = G \cap A = F \cap A$$

By connectedness of A,

$$S \cap A = \phi \quad \text{or} \quad S \cap A = A$$

$$\begin{array}{ccc} S \cap A = \phi & & S \cap A = A \\ \| & & \| \\ G \cap A & & F \cap A \end{array}$$

$\therefore A \subset \underbrace{X \setminus G}_{\text{closed}}$ $\qquad$ $\therefore A \subset \underbrace{F}_{\text{closed}}$

$$\therefore \bar{A} \subset X \setminus G \qquad \therefore \bar{A} \subset F$$

$$\begin{array}{ccc} \bar{A} & & \bar{A} \\ \cup & & \cup \\ B & & B \end{array}$$

$$\therefore S = G \cap B = \phi \quad \text{or} \quad \therefore S = F \cap B = B$$

Exercise. Use this result to prove that every connected component is closed.

Remark. This result is always useful to extend connectedness to a closure.

04-11-p5

Tuesday, April 11, 2017 11:07 AM

Theorem. An infinite product of connected spaces is also connected.

Idea of proof. Let us review the case for finite product.

$$\left.\begin{aligned} X\times\{b\} &= \pi_Y^{-1}(b) \\ \{a\}\times Y &= \pi_X^{-1}(a) \end{aligned}\right\} \text{ are connected}$$

$$A_\alpha = \pi_X^{-1}(a) \cup \pi_Y^{-1}(b)$$

all possibilities of α

For $P = \prod_{\lambda\in I} X_\lambda$, we take

$$A_\alpha = \bigcup_{k=1}^{\text{finite}} \pi_{\lambda_k}^{-1}(a_\lambda)$$

More possible combinations of α

One can also prove that

$$A_\alpha \cap A_\beta \neq \emptyset \quad \forall\ \alpha, \beta$$

However, $P \neq \bigcup_\alpha A_\alpha$

Luckily, $P = \overline{\bigcup_\alpha A_\alpha}$ $\begin{cases} \text{true in } \mathcal{T}_\Pi \\ \text{false in } \mathcal{T}_{\text{BOX}} \end{cases}$